Quantum correlations III

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- Rules of Quantum Mechanics:

- Let us begin this lecture with the citation taken from the Omnés book:

- "*Every physical system, whatever an atom or a star is assumed to be described by a universal kind of mechanic which is Quantum Mechanics*".

- Thus, one should quantize both probability calculus and the concept of (classical) composite systems.

- We remind that we want to understand , among other things, the name: quantum correlations.

- It is desirable to list the rules of quantum theory.

- **Rule 1.** *The theory of given individual isolated physical system can be entirely expressed in terms of a specific Hilbert space and mathematical notions associated with it, particularly a specific algebra of operators.*

- **Rule 2.** *A specific self-adjoint operator is associated with an* **isolated** *physical system. This operator is the system Hamiltonian H. It determines Heisenberg's dynamics (which replace Newton's law of motion). The dynamics is expressed by a continuous 1-parameter unitary group of operators $U(t)$, $t \in \mathbb{R}$ having the Hamiltonian H as its infinitesimal generator. In particular, the evolution of an observable O is given by $O_t = U(t)OU^*(t) \equiv \alpha_t(O)$ (in Heisenberg picture) while the evolution of a state is given by $\Psi_t = U(t)\Psi$ (in Schrödinger picture).*

- **Rule 3.** *On the specific algebra $\mathfrak{A}$ of operators associated with a given physical system there exist a family of linear positive normalized forms on $\mathfrak{A}$. They form the set of states $\mathfrak{S}$. The interpretation of a state φ ($\varphi \in \mathfrak{S}$, $\mathfrak{A} \ni O \to \varphi(O) \in \mathbb{C}$) as given by Born is that the number $\varphi(O)$ (real if φ and O are self-adjoint) is the expectation value of observable O.*

- and:

- **Rule 4.** *Let two physical systems S_1 and S_2 be represented by Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, algebras $\mathfrak{A}_1$, $\mathfrak{A}_2$, sets of states $\mathfrak{S}_1$, $\mathfrak{S}_2$ and finally Hamiltonians H_1, H_2 respectively. When they are combined in one (composite) system S then its Hilbert space $\mathcal{H}$ is equal $\mathcal{H}_1 \otimes \mathcal{H}_2$ and its algebra $\mathfrak{A} = \mathfrak{A}_1 \otimes \mathfrak{A}_2$. When the systems S_1 and S_2 are dynamically independent, Hamiltonian associated with the composite system is given by $H = H_1 \otimes \mathbb{1} + \mathbb{1} \otimes H_2$.*

- and finally

- **Rule 5.** *Let a composite system* $(\mathfrak{A} \equiv \mathfrak{A}_1 \otimes \mathfrak{A}_2, \mathfrak{S}_{\mathfrak{A}}, \alpha_t(\cdot))$ *be given. When, one is interested only in time evolution of a subsystem, say that labeled by* "1"*, then a reduction of (global) Hamiltonian type dynamics should be carried out. As a result, time evolution of the subsystem* "1" *is described by a one parameter family of (linear) maps* $T_t : \mathfrak{A}_1 \to \mathfrak{A}_1$ *such that* $T_t(f) \geq 0$ *for any* t *and a positive* $f \in \mathfrak{A}_1$ *(so positivity preserving), and* $T_t(\mathbb{1}) = \mathbb{1}$*.*

- To comment the last rule we add

- **Remark 6.**
  - *If one adds Markovianity assumption then* $\{T_t\}$ *is also a semigroup, i.e.* $T_t \circ T_s = T_{t+s}$ *for non-negative* t *and* s*.*
  - *Frequently, more stronger assumption on positivity -* $complete\ positivity$ *- is relevant.*

- **Example 7.** *(Dirac's formalism)*

1. *(Dirac's Quantum Mechanics.) A separable, infinite dimensional Hilbert space $\mathcal{H}$ is associated with a system. The specific algebra $\mathfrak{A}$ is taken to be $B(\mathcal{H})$ - the set of all linear, bounded operators on $\mathcal{H}$. The set of states is determined by density matrices, i.e. positive trace class operators with trace equal to 1. The expectation value $< A >$ of $A = A^* \in B(\mathcal{H})$ at a state ϱ is given by $< A >= \mathrm{Tr}\varrho A$ where ϱ is a density matrix.*
2. *(composite system in Dirac's formalism) Let S_i be a physical system, $\mathcal{H}_i$ be an associated Hilbert space, $B(\mathcal{H}_i)$ be an associated algebra and $\mathfrak{S}_i$ denote set of states (density operators) for $i = 1, 2$. Then Hilbert space, associated algebra and set of states for composite system are given by $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, $B(\mathcal{H}_1 \otimes \mathcal{H}_2) \approx B(\mathcal{H}_1) \otimes B(\mathcal{H}_2)$ and $\mathfrak{S}$ where, in general contrary to the classical case, one has $\mathfrak{S} \neq \overline{conv}(\mathfrak{S}_1 \otimes \mathfrak{S}_2)$!!!*

- and we wish to point out that the just given scheme also contains the classical physics!

- **Example 8.** *(Classical systems)*

  1. *(classical system) Assume that $\mathfrak{A}$ is an abelian C*-algebra with unit $\mathbb{1}$. Then (according to Gelfand-Neimark theorem (see eg. Sakai's book) $\mathfrak{A}$ can be identified with the C^*-algebra (see below for the definition of C^*-algebra) of all complex valued continuous functions on Γ, where Γ is a compact Hausdorff space. Hence, a state (normalized, positive, linear form) on $\mathfrak{A}$ leads to a probability measure on Γ (as it was pointed out in the last lecture). Consequently, the probabilistic scheme described in last lecture was recovered.*

and

2. *(classical composite system) Take (for $i = 1, 2$) two abelian C^*-algebras $\mathfrak{A}_i$ with unit and combine them in one system (cf Rule 4.4). Then we have $\mathfrak{A} \equiv \mathfrak{A}_1 \otimes \mathfrak{A}_2 = C(\Gamma_1) \otimes C(\Gamma_2) \approx C(\Gamma_1 \times \Gamma_2)$. Take a state (normalize, linear, positive form) on $\mathfrak{A}$. Then, using the last lecture, there exists a probability measure on $\Gamma_1 \times \Gamma_2$. Thus, we recover a classical composite system described in the previous section. In particular, (as it was said at the end of the previous lecture) one has $\mathfrak{S} = \overline{conv}(\mathfrak{S}_1 \otimes \mathfrak{S}_2)$!*

- To comment the just presented Rules as well as to make clear "*specific algebra of operators*" (Rule 1) we need some definitions.

- **Definition 9.** *Let $\mathfrak{A}$ be a Banach space. We say that $\mathfrak{A}$ is a Banach algebra if it is an algebra, i.e. a multiplication*

$$\mathfrak{A} \times \mathfrak{A} \ni \langle f, g \rangle \mapsto fg \in \mathfrak{A} \tag{1}$$

is defined in such way that for every $f, g \in \mathfrak{A}$ one has

$$\|fg\| \leq \|f\| \, \|g\| \tag{2}$$

It follows that multiplication in Banach algebra is separately continuous in each variable.

- **Definition 10.** *An involution on an algebra $\mathfrak{A}$ is a antilinear map $f \to f^*$ such that for all $f, g \in \mathfrak{A}$ and $\lambda \in \mathbb{C}$ one has*

$$(f^*)^* = f \tag{3}$$

$$(fg)^* = g^* f^* \tag{4}$$

$$(\lambda f)^* = \overline{\lambda} f^* \quad \text{and} \quad (f+g)^* = f^* + g^* \tag{5}$$

A $*$-algebra is an algebra with involution. A $*$-Banach algebra $\mathfrak{A}$ is a $*$-algebra such that $\mathfrak{A}$ is a Banach algebra and $\|f^*\| = \|f\|$.

Definition 11. *A C*-algebra $\mathfrak{A}$ is a $*$-Banach algebra such that the norm $\|\cdot\|$ satisfies*

$$\|ff^*\| = \|f\|^2 \tag{6}$$

- Finally:

- **Definition 12.** *A C^*-algebra $\mathfrak{M}$, acting on a Hilbert space $\mathcal{H}$, that is closed in the weak operator topology and contains the unit $\mathbb{1}$ is said to be a von Neumann algebra (or equivalently, a W^*-algebra).*

- **Example 13.** *1. $M_n(\mathbb{C})$ - all $n \times n$ matrices with complex entries.*
  2. Denote by $\mathcal{B}(\mathcal{H})$ the family of all bounded linear operators on a Hilbert space $\mathcal{H}$. $\mathcal{B}(\mathcal{H})$ is a von Neumann algebra.

- It is worth pointing out that a scheme based on $\mathcal{B}(\mathcal{H})$ (Dirac's formalism) is not able to describe all quantum systems!!!

- Moreover, there is also a warning (Winter, Phys. Rev. **71**, 737 (1947)).

- **Proposition 14.** *It is impossible to find two elements a, b in a Banach algebra $\mathfrak{A}$ such that*
$$ab - ba = 1. \tag{7}$$

- The principal significance of Proposition 14 stems from the following conclusion: *it is impossible to realize canonical commutation relations in terms of a Banach algebra*.

- So, it is impossible to carry out a canonical quantization on finite dimensional spaces.

- Consequently, **it is difficult to speak about quantumness of finite dimensional systems**.

- Now we are in a position to comment the above listed Rules for quantization.

- Firstly, the specific algebra mentioned in Rule 1 means the collection of bounded functions of, in general, unbounded operators.

- Note that such operators appear in the theory due to the procedure of quantization. On the other hand, Proposition 14 clearly shows that finite dimensional models are not able to describe genuine quantum systems. Therefore, they can only provide so called "toy" models!

- Rule 3 is saying that a (quantum) observable is a non-commutative counterpart of a stochastic variable.

- In particular, this implies that quantum probability should appear.

- The standard form of non-commutative probability calculus is provided by the pair $(\mathfrak{A}, \varphi)$, where $\mathfrak{A}$ is a C^*-algebra, φ is a state, i.e. a linear functional on $\mathfrak{A}$ such that $\varphi(\mathbb{1}) = 1$, and $\varphi(a) \geq 0$ for all $a \geq 0$.

- It is important to note here that on the additional assumption that $\mathfrak{A}$ is abelian one can recover the classical probability scheme.
- Namely, *Gelfand-Naimark theorem*, any abelian C^*-algebra $\mathfrak{A}$ with unit $\mathbb{1}$ can be identified with the collection of all complex valued functions defined on a compact Hausdorff space E.
- This implies the existence of a probability measure μ on X, which is uniquely determined by a state.
- Consequently, fundamentals of a (classical) probability were obtained.
- But, as it is impossible to embed a non-commutative C^*-algebra into commutative one, there is no hope to embed quantum probability into larger classical probability scheme.

- Consequently, there is not "room" for hidden variable models
- There is also another extremely important motivation for more "refined" algebras - for W^*-algebras.
- Namely, classical mechanics demands the differential and integral calculus for its description.
- It is natural to expect that quantum mechanics demands non-commutative calculus for its description.
- This is the case! ($functional\ analysis$).
- Consequently, to simplify our exposition of quantum rules, by a *specific algebra* we will mean either C^*-algebra or W^*-algebra.

- The Rule 4 is saying that to form a bigger system consisting of two smaller subsystems, a tensor product of appropriate algebras should be taken.

- As this concept for **Banach spaces is not trivial one**, we will clarify this notion in the next lecture.

- We wish to close this lecture with remarks concerning the following question: Why we do not restrict ourselves to Dirac's formalism only?

- In other words, why we will not restrict ourselves to formalism specified in Example 7?

- The answer follows from the following observations (see, for example R. Haag book: *Local Quantum Physics*):

- Individual features of a system as well as a relation between a system and the region occupied by this system should be taken into account, (examples will be provided in next lectures)

- To take into account a causality one should have a possibility to say how far away is a subsystem S_1 from S_2 (in classical case, subsystems are described by subspaces of an Euclidean space. So this question has an easy solution).

- The above question, for quantum case, demands more general setting than that one which is offered by Example 7. Thus, Dirac's formalism is not enough!

- Quantum field theory demands more general approach than that given by Dirac's formalism, see for example the above mentioned R. Haag book!

- The important point to note here is that quantum correlations as a phenomenon was observed in quantum field theory many decades ago.

- So, not only in “atomic physics” and in Quantum Information.

- Probably, the best example is given by the Reeh and Schlieder theorem. This theorem is saying that any state vector of a quantum field can be approximated by an action of “local” operator acting on the vacuum. To this end the operator must exploit the small but non zero long distant correlations which exist in the vacuum.

- Other examples of quantum correlations in quantum field theory can also be found.

- Examples of quantum systems:

- Lattice models, solid body physics:

- **Example 15.** *UHF - uniformly hyperfinite algebra.*
  $\mathbb{Z}^n$ *is the Cartesian product of integer numbers* $\mathbb{Z}$ *with* $n = \{1,\ 2,\ 3,\ \ldots N\}$. *Let* $\alpha \in \mathbb{Z}^n$ *be an arbitrary, fixed site of the lattice. With each site* α *we associate a Hilbert space* $\mathcal{H}_\alpha$. *It is required that each Hilbert space* $\mathcal{H}_\alpha$ *is finite dimensional. The Hilbert space associated with a finite subset* $\Lambda \subset Z^n$ *is given by* $\mathcal{H}_\Lambda = \otimes_{\alpha\in\Lambda}\mathcal{H}_\alpha$. *We put* $\mathfrak{A}_\Lambda = B(\mathcal{H}_\Lambda)$. *The algebra of operators associated with the whole space* $\mathbb{Z}^n$ *is equal to* $\mathfrak{A} = \overline{\bigcup_\Lambda \mathfrak{A}_\Lambda}$.

- The preceding example can be generalized.

- **Example 16.** $\textit{Quasi local algebras.}$
  Replace $\mathbb{Z}^n$ *by* $\mathbb{R}^n$*, i.e we replace integer numbers* $\mathbb{Z}$ *by real numbers* $\mathbb{R}$*. In physical terms it means that now we are interested in continuous systems. We associate a Hilbert space* $\mathcal{H}_\Lambda$ *(usually infinite dimensional) with the region (bounded subset)* $\Lambda \subset \mathbb{R}^n$*. Put* $\mathfrak{A}_\Lambda = B(\mathcal{H}_\Lambda)$*. It means that with a region* Λ *in* $\mathbb{R}^n$ *we associate* $local$ *observables and these observables generate the (specific) algebra* $\mathfrak{A}_\Lambda$*. For* $\Lambda \subset \Lambda'$ *we have* $\Lambda' = \Lambda \cup (\Lambda' \setminus \Lambda)$ *and according to Rule 4.4 one has* $\mathcal{H}_{\Lambda'} = \mathcal{H}_\Lambda \otimes \mathcal{H}_{\Lambda' \setminus \Lambda}$*. Then, the algebra of all observables is given by* $\mathfrak{A} = \overline{\bigcup_\Lambda \mathfrak{A}_\Lambda}$*.*

- Note: observables are localized!

- BUT, this does not mean that states are localized!

- Such models are basic for Quantum Statistical Physics.

- To say more on such models we need:

Definition 17. *A state $\omega \in \mathfrak{S}_{\mathfrak{A}}$ is called locally normal if $\omega|_{\mathfrak{A}_{\Lambda}}$ is normal for every Λ.*

- Thus, a state is locally normal, if its restriction to an algebra associated with any bounded region is given by a (local) density matrix.

- Usually, we are interested in locally normal states (related to stability of the matter.

- Example 16 still can be generalized. The motivation of that follows from Rule 1: observables of a system associated with a region $\Lambda \subseteq \mathbb{R}^n$ do not need to generate the algebra $B(\mathcal{H}_\Lambda)$. We emphasize that this problem is in "the heart" of Haag-Kastler approach to Quantum Field Theory!

- **Example 18.** *Let $\mathbb{R}^n$ be a "coordinate" space of a field and $\Lambda \subset \mathbb{R}^n$ be a region. With the region Λ we associate a C*-algebra $\mathfrak{A}_\Lambda$ (which is not necessary of the form $B(\mathcal{H}_\Lambda)$). However, we are assuming that additional conditions, which will discussed in the next lecture, hold for the family $\{\mathfrak{A}_\Lambda\}_{\Lambda\subset\mathbb{R}^n}$. Put $\mathfrak{A} = \overline{\bigcup_\Lambda \mathfrak{A}_\Lambda}$.*